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A Combinatorial Approach to the Solitaire Game

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SUMMARY The classical game of peg solitaire has uncertain origins, but was certainly popular by the time of Louis XIV, and was described by Leibniz in 1710. One of the classical problems concerning peg solitaire is the feasibility issue. An early tool used to show the infeasibility of various peg games is the *rule-of-three* [Suremain de Missery 1841]. In the 1960s the description of the *solitaire cone* [Boardman and Conway] provides necessary conditions: valid inequalities over this cone, known as pagoda functions, were used to show the infeasibility of various peg games. In this paper, we recall these necessary conditions and present new developments: the *lattice criterion*, which generalizes the rule-of-three; and results on the strongest pagoda functions, the facets of the solitaire cone.
key words: *solitaire peg game, feasibility, combinatorial approach*

1. Introduction and Basic Definitions

1.1 Introduction

Peg solitaire is a peg game for one player which is played on a board containing a number of holes. The most common modern version uses a cross shaped board with 33 holes—see Fig. 1—although a 37 hole board is common in France. Computer versions of the game now feature a wide variety of shapes, including rectangles and triangles. Initially the central hole is empty, the others contain pegs. If in some row (column respectively) two consecutive pegs are adjacent to an empty hole in the same row (column respectively), we may make a *move* by removing the two pegs and placing one peg in the empty hole. The objective of the game is to make moves until only one peg remains in the central hole. Variations of the original game, in addition to being played on different boards, also consider various alternate starting and finishing configurations.

The game itself has uncertain origins, and different legends attest to its discovery by various cultures. An authoritative account with a long annotated bibliography can be found in the comprehensive book of Beasley [3]. The book mentions an engraving of Berey, dated 1697, of a lady with a solitaire board. The book also contains a quotation of Leibniz [6] which was written for the Berlin Academy in 1710. Apparently the first theoretical study of the game that was published was done in 1841 by Suremain de Missery, and was reported in a paper by Vallot [8]. The modern mathematical study of the game dates to the 1960s at Cambridge University. The group was led by Conway who has written a chapter in [4] on various mathematical aspects of the subject. One of the problems studied by the Cambridge group is the following basic *feasibility* problem (see Definition 1 in the sequel for a formal definition):

Peg solitaire feasibility problem: Given a board B and a pair of configurations (c, c') on B, determine if the pair (c, c') is *feasible*, that is, if there is a legal sequence of moves transforming c into c'.

The complexity of the feasibility problem for the game played on a n by n board was shown by Uehara and Iwata [7] to be NP-complete, so easily checked necessary and sufficient conditions for feasibility are unlikely to exist. In this paper, we recall constructions used to prove the infeasibility of some pair (c, c'): the *rule-of-three* in Sect. 2, the *solitaire cone* in Sect. 3.1; and present new developments in Sects. 3.2 and 4.

1.2 Basic Definitions

In this subsection we introduce some terminology used throughout this paper. The *board* of a peg solitaire game is a *finite* subset $B \subset \boldsymbol{Z}^2$. Thus, B stands for the set of locations (i, j) of holes of the board on which the game is played. For example, the classical 33-board is: $B = \{(i, j)\colon -1 \le i \le 1,\ -3 \le j \le 3\} \cup \{(i, j) : -3 \le i \le 3,\ -1 \le j \le 1\}$. A *configuration* c on the board is an integer vector $c \in \boldsymbol{Z}^B \subset \boldsymbol{R}^B$. It can be interpreted as a configuration of pegs on the board: in the usual game, all configurations c lie in $\{0, 1\}^B$, with the interpretation that hole $(i, j) \in B$ contains a peg if $c_{i,j} = 1$ and is empty if $c_{i,j} = 0$; extending this, we allow any integer (possibly negative) number $c_{i,j}$ of pegs to occupy any hole $(i, j) \in B$. The *complement* of a $\{0, 1\}$-configuration $c \in \{0, 1\}^B$ is defined to be the configuration $\bar{c} := \boldsymbol{l} - c$ where $\boldsymbol{l} = (1, 1, \dots, 1) \in \boldsymbol{R}^B$ is the all-ones configuration. A *move* or a *jump* μ is a vector in $\boldsymbol{R}^B$ which has 3 non-zero entries: two entries of -1 in the positions from which pegs are removed and one entry of 1 for the hole receiving the new peg.

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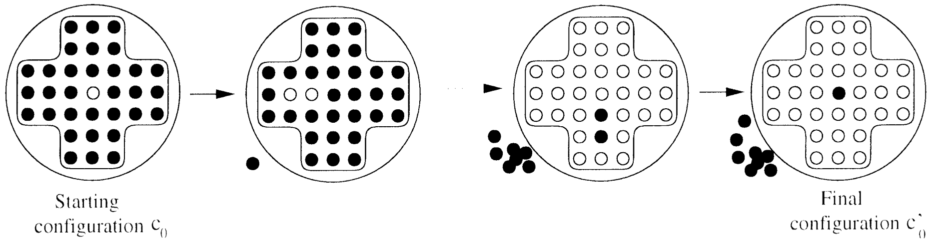

Fig. 1 A feasible English solitaire peg game with possible first and last moves.

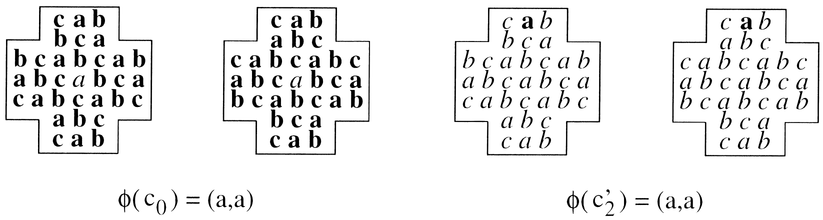

Fig. 2 The score of a final configuration with only one peg remaining.

We can now make the peg solitaire feasibility problem precise.

Definition 1.1: Given a board B and an associated set of moves $\mathcal{M}$, a pair (c, c') of configurations is *feasible* if there is a sequence $\mu^1, \ldots, \mu^k \in \mathcal{M}$ of moves on B such that

$$c' - c = \sum_{i=1}^{k} \mu^i \text{ and } c + \sum_{j=1}^{i} \mu^j \in \{0,1\}^B \text{ for } i = 1, \ldots, k$$

For instance, the English 33-board admits 76 moves (none over the 8 corners, 24 moves over the 12 holes next to a corner and 52 moves over the 13 remaining holes); see Fig. 1 for possible first and last moves in some sequence of moves transforming the initial configuration c_0 to its complementary c'_0.

2. The Rule-of-Three

In this section we recall the so-called rule-of-three (cf. [3], [4]), a classical construction used to test solitaire game feasibility. The rule-of-three can be used, for example, to show that on the cross shaped English 33-board, starting with the initial configuration c_0 of Fig. 1, the only reachable final configurations with *exactly one* peg are c'_0 (given in Fig. 1), c'_1, c'_2, c'_3 and c'_4 with, respectively, a final peg in position $(0,0), (-3,0), (0,3), (3,0)$ and $(0,-3)$.

Let $\boldsymbol{Z}_2 := \{a, b, c, e\}$ be the Abelian group with identity e and addition table $a + a = b + b = c + c = e$, $a + b = c$, $a + c = b$, $b + c = a$. Define the following two maps $g_1, g_2 : \boldsymbol{Z}^2 \longrightarrow \boldsymbol{Z}_2$, which simply color the integer lattice $\boldsymbol{Z}^2$ by diagonals of a, b and c in either direction; see Fig. 2:

$$g_1(i,j) := \begin{cases} a & \text{if } (i+j) \equiv 0 \pmod 3 \\ b & \text{if } (i+j) \equiv 1 \pmod 3 \\ c & \text{if } (i+j) \equiv 2 \pmod 3 \end{cases},$$

$$g_2(i,j) := \begin{cases} a & \text{if } (i-j) \equiv 0 \pmod 3 \\ b & \text{if } (i-j) \equiv 1 \pmod 3 \\ c & \text{if } (i-j) \equiv 2 \pmod 3 \end{cases}.$$

For each $(i,j) \in B \subset \boldsymbol{Z}^2$ let $e_{i,j}$ be the (i,j)th unit vector in $\boldsymbol{R}^B$, and define the *score map* to be the $\boldsymbol{Z}$-module homomorphism $\phi : \boldsymbol{Z}^B \longrightarrow \boldsymbol{Z}_2^2$ with $\phi(e_{i,j}) := \big(g_1(i,j)\ ,\ g_2(i,j)\big)$. Thus, the score of a configuration $c \in \boldsymbol{Z}^B$ is given by

$$\phi(c) \quad = \quad \sum_{(i,j)\in B} c_{i,j} \cdot \big(g_1(i,j)\ ,\ g_2(i,j)\big).$$

Since the board B under discussion will always be clear from the context, we use the notation ϕ for any board. For instance, the score of the configuration c'_0 of one peg in the center of the English 33-board is $\phi(c'_0) = (a,a)$, as is also the score of its complement c_0; see Fig. 2. The score of the board B (all holes filled) is defined to be $\phi(B) = \phi(\boldsymbol{l})$. It is easy to verify that any feasible move μ on any board B has the identity score $\phi(\mu) = (e,e)$. This gives the following proposition.

Proposition 2.1 (The rule-of-three):
A necessary condition for a pair of configuration (c, c') to be feasible is that $\phi(c' - c) = (e,e)$, namely, $c' - c \in \operatorname{Ker}(\phi)$.

Using Prop. 2.1, we can show that, besides the configuration c'_0 given in Fig. 1, the only final configuration c' with exactly one non-zero entry $c'_{i,j} = 1$ forming a feasible pair (c_0, c') are the 4 configurations c'_1, c'_2, c'_3 and c'_4. Figure 2 shows that $\phi(c') = (a,a) = \phi(c_0)$ if c'

is one of $c'_0, c'_1 \ldots c'_4$, whereas $\phi(c') \neq (a, a)$ otherwise.

With $\bar{c} = \boldsymbol{l} - c$ the complement of c, the rule-of-three implies that $(c, \bar{c})$ is feasible only if $\phi(\bar{c}) = \phi(c)$, which is equivalent to $\phi(B) = \phi(c) + \phi(\bar{c}) = \phi(c) + \phi(c) = (e, e)$. In other words, a necessary condition for the configurations pair $(c, \bar{c})$ to be feasible is that the board score is $\phi(B) = (e, e)$. Such a board is called a *null-class* board in [3]. For example, the score of the English 33-board is $\phi(B) = \phi(c_0) + \phi(c'_0) = (a, a) + (a, a) = (e, e)$.

3. Solitaire Cone and Pagoda Functions

3.1 Solitaire Cone

A first relaxation of the feasibility problem is to allow any integer (positive or negative) number of pegs to occupy any hole for any *intermediate* configurations. We call this game the *integer game*, and call the original game the *0-1 game*. Note that in a 0-1 game we require that for each intermediate configuration of the game a hole is either empty or contains a single peg. Clearly, c', c is integer feasible if and only if

$$c' - c \in IC_B = \left\{ \sum_{\mu \in \mathcal{M}} \lambda_\mu \mu : \lambda_\mu \in \boldsymbol{N} \right\}$$

where the *integer solitaire cone* IC_B is the set of all *non-negative integer* linear combinations of moves. Unfortunately deciding if $c' - c$ can be expressed as the sum of move seems to be a hard computational problem. We get the following necessary criterion:

Proposition 3.1 (The integer cone criterion):
A necessary condition for a pair of configurations (c, c') to be feasible is that $c' - c \in IC_B$.

A further relaxation of the game leads to a more tractable condition. In the *fractional game* we allow any fractional (positive or negative) number of pegs to occupy any hole for any *intermediate* configurations. A *fractional move* is obtained by multiplying a move by any positive scalar and is defined to correspond to the process of adding a move to a given configuration. For example, let $c = [111]$, $c' = [101]$. Then $c' - c = [0 {-1} 0] = \frac{1}{2}[-1 -1 1] + \frac{1}{2}[1 -1 -1]$ is a feasible fractional game and can be expressed as the sum of two fractional moves, but is not feasible as a 0-1 or integer game. Clearly, c', c is fractional feasible if and only if

$$c' - c \in C_B = \left\{ \sum_{\mu \in \mathcal{M}} \lambda_\mu \mu : \lambda_\mu \in \boldsymbol{R}^+ \right\}$$

where the *solitaire cone* C_B is the set of all *non-negative* linear combinations of moves. We get the weaker, but useful, following necessary criterion:

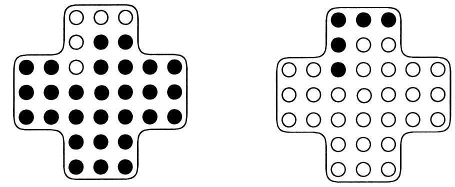

Fig. 3 An infeasible classical solitaire peg game.

Proposition 3.2 (The cone criterion):
A necessary condition for a pair of configurations (c, c') to be feasible is that $c' - c \in C_B$.

The condition $c' - c \in C_B$ is therefore a necessary condition for the feasibility of the original peg game and, more usefully, provides a certificate for the infeasibility of certain games. The certificate of infeasibility is any inequality valid for C_B which is violated by $c' - c$. According to [3], page 71, these inequalities "were developed by J.H. Conway and J.M. Boardman in 1961, and were called *pagoda functions* by Conway ...." They are also known as *resource counts*, and are discussed in some detail in Conway [4]. The strongest such inequalities are induced by the facets of C_B. For example, the facet (iii) (given by Beasley) of Fig. 4 induces an inequality $a \cdot x \leq 0$ that is violated by $c' - c$ with (c, c') given in Fig. 3: $(c' - c) \cdot a = 2 > 0$. This implies that this game is not feasible even as a fractional game and, therefore, not feasible as an integer game or classical 0-1 game either.

3.2 Facets of the Solitaire Cones

Most of the results can be applied to boards which are subsets of the square lattice in the plane, such as the original peg solitaire board. For simplicity let us consider rectangular boards. For $n \geq 4$ or $m \geq 4$, the solitaire cone $C_{B_{m,n}}$ associated to the m by n board $B_{m,n}$ is a pointed full-dimensional cone and the moves of the solitaire cone are extreme rays; see [1] for a detailed study of $C_{B_{m,n}}$. Let us assume that $m \geq 4$ or $n \geq 4$ and that the holes of $B_{m,n}$ are ordered in some way. It is convenient to display $c' - c$ and μ as m by n matrices, although of course all products should be interpreted as dot products of the corresponding mn-vectors. We represent the coefficients of the facet inducing inequality $az \leq 0$ by the m by n array $a = [a_{i,j}]$. It is a convenient abuse of terminology to refer to a as a *facet* of C_B. A *corner* of a is a coefficient $a_{i,j}$ with $i \in \{1, m\}$ and $j \in \{1, n\}$. The notation $\mathcal{T} = (t_1, t_2, t_3)$ refer to a *consecutive triple* of row or column indices. Each consecutive triple defines a *triangle inequality*: $a_{t_1} \leq a_{t_2} + a_{t_3}$ and a triangle inequality is *tight* if equality holds. The following theorem summarizes known results on properties of valid inequalities (pagoda functions) for C_B.

Theorem 3.3[3]: For each valid inequality $a = [a_{i,j}]$

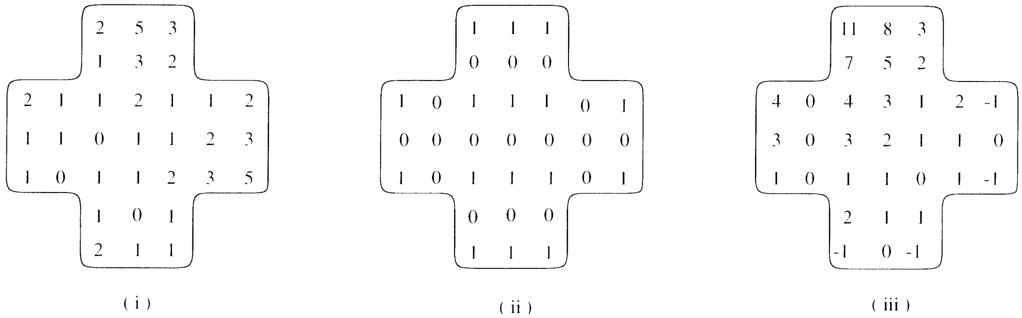

Fig. 4 Three facets of the English solitaire cone.

for C_B

1. For every consecutive triple $\mathcal{T} = (t_1, t_2, t_3)$, the associated triangle inequality must hold.
2. Negative coefficients of a can only occur in corners.
3. If $\mathcal{T} = (t_1, t_2, t_3)$ is a consecutive triple with $a_{t_2} = 0$ then $a_{t_1} = a_{t_3}$.
4. If two consecutive row (respectively, column) entries of a are zero the entire row (respectively, column) is zero.

It is not feasible to generate all facets for reasonably sized boards, and in general no characterization of facets is known. A large class of facets can, however, be generated by the following procedure.

Genfacet /*procedure to generate a facet matrix a of C_B */

1. Choose a proper subset of coefficients of a satisfying: (a) If a corner is chosen, all coefficients in the row and/or column of at length at least 4 containing the corner must also be chosen; and (b) If two consecutive coefficients are chosen, their entire row and column must also be chosen. Set these chosen coefficients to zero.
2. Choose any undefined coefficient that is not a corner and set it to one.
3. Choose a consecutive triple $\mathcal{T} = (t_1, t_2, t_3)$ for which precisely two of the corresponding coefficients of a are defined. Define the remaining coefficient by the associated tight triangle inequality providing this does not violate any other triangle inequality for a.
4. Repeat step 3 until no further coefficient of a can be defined.

Theorem 3.4[1]**:** Given an m by n board B, with $m \geq 4$ or $n \geq 4$, if Genfacet terminates with all elements of a defined, then a is a facet of C_B.

Genfacet can easily be adapted to non-rectangular boards that are connected subsets of the square grid, such as the original peg solitaire game. The notion of corner generalizes in the obvious way to all holes that have exactly one horizontal and vertical neighbour. For example, the original English game has 8 corners and by Genfacet we can generate the facets (i) and (ii), but not facet (iii), given in Fig. 4. Interesting applications of Genfacet include a characterization of 0-1 facets and exponential upper and lower bounds on the number of facets (in the dimension $|B|$ of C_B), see [1] for details and other geometric and combinatorial properties of C_B.

4. The Lattice Criterion

4.1 The Solitaire Lattice

For the fractional game, we relaxed the integrality while keeping the non-negativity condition. Another relaxation of the integer game is to drop the non-negativity while keeping the integrality. It amounts, besides allowing any integer (positive or negative) number of pegs for any intermediate configurations, to allow *additive moves*. The configuration of an additive move μ^+ (jumping over an empty hole and putting a peg in) is $c_\mu^+ = -c_\mu$ where c_μ is the configuration of an ordinary (subtractive) move μ. We call this game the *lattice game*. Clearly, c', c is *lattice* feasible if and only if

$$c' - c \in L_B = \left\{ \sum_{\mu \in \mathcal{M}} \lambda_\mu \mu : \lambda_\mu \in \boldsymbol{Z} \right\}$$

where the *solitaire lattice* L_B is the set of all *integer* linear combinations of moves. It gives the following criterion (weaker than Prop. 3.1):

Proposition 4.1 (The lattice criterion):
A necessary condition for a pair of configurations (c, c') to be feasible is that $c' - c \in L_B$.

Since the score ϕ is a homomorphism of $\boldsymbol{Z}$-modules which maps each lattice generator $\mu \in \mathcal{M}$ to (e, e), it follows that $\phi(v) = (e, e)$ for any $v \in L_B$; i.e., for any board B and any pair (c, c') on B if $c' - c \in L_B$ then $c' - c \in \mathrm{Ker}(\phi)$. In other words, as stated in the following proposition, the lattice criterion is generally stronger than the rule-of-three.

Proposition 4.2: For any board B, we have $L_B \subseteq \mathrm{Ker}(\phi)$.

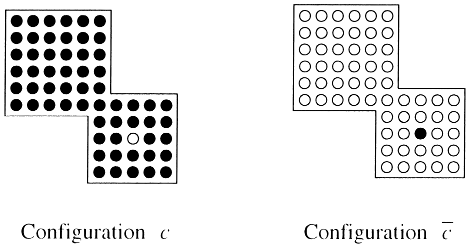

Fig. 5 An infeasible game satisfying the rule-of-three but not the solitaire lattice criterion.

Figure 5 provides an example of a null-class board and a game on it whose associated pair $(c, \bar{c})$ satisfies $\bar{c} - c \in \mathrm{Ker}(\phi)$ but $\bar{c} - c \notin L_B$ (see Prop. 4.5). This shows that the lattice criterion may be strictly stronger than the rule-of-three and therefore could be more useful in proving infeasibility.

4.2 The Lattice Criterion versus the Rule-of-Three

The solitaire lattice of a board B is typically of full rank $|B|$ and, in this case, let $\det(L_B)$ denote his determinant. Since $L_B \subseteq \mathrm{Ker}(\phi)$ for any board B, we have $\boldsymbol{Z}^B/\mathrm{Ker}(\phi) \subseteq \boldsymbol{Z}^B/L_B$ and therefore, $|\boldsymbol{Z}^B/\mathrm{Ker}(\phi)| \leq |\boldsymbol{Z}^B/L_B|$. For the usual board $|\boldsymbol{Z}^B/\mathrm{Ker}(\phi)| = 16$: one can easily check that the 16 possible $\{0,1\}$-configurations on a 2×2 board are mapped by ϕ precisely onto the 16 elements of $\boldsymbol{Z}_2^2$. In other words, if a board B contains a 2×2 sub-board then $|\boldsymbol{Z}^B/\mathrm{Ker}(\phi)| = |\mathrm{Im}(\phi)| = 16$. For a typical board B, the map ϕ is onto and the lattice L_B is full rank, giving $|\boldsymbol{Z}^B/\mathrm{Ker}(\phi)| = 16$ and $|\boldsymbol{Z}^B/L_B| = \det(L_B)$. This gives the following useful lemma.

Lemma 4.3: For a board B such that ϕ is an onto map and L_B is full rank, $L_B = \mathrm{Ker}(\phi)$ if and only if $\det(L_B) = 16$.

4.3 The Lattice Criterion Beats the Rule-of-Three

A close study of $L_{B_{m,n}}$ provides a canonical basis (which is precisely the *Hermite basis*) for this lattice, see [5] for details. We have:

Theorem 4.4: Let $B_{m,n}$ be any $m \times n$ board with $n \geq 4$ or $m \geq 4$. The solitaire lattice $L_{B_{m,n}}$ has full rank with determinant $\det(L_{B_{m,n}}) = 16$, hence $L_{B_{m,n}} = \mathrm{Ker}(\phi)$. $L_{B_{m,n}}$ is characterized by $c \in L_{B_{m,n}}$ if and only if

$$\begin{aligned}
&\sum\{c_{i,0} : 0 \leq i \leq n-1, i \not\equiv 0 (\mathrm{mod}\ 3)\} \equiv 0 \ (\mathrm{mod}\ 2)\\
&\sum\{c_{i,0} : 0 \leq i \leq n-1, i \not\equiv 1 (\mathrm{mod}\ 3)\} \equiv 0 \ (\mathrm{mod}\ 2)\\
&\sum\{c_{i,1} : 0 \leq i \leq n-1, i \not\equiv 0 (\mathrm{mod}\ 3)\} \equiv 0 \ (\mathrm{mod}\ 2)\\
&\sum\{c_{i,1} : 0 \leq i \leq n-1, i \not\equiv 1 (\mathrm{mod}\ 3)\} \equiv 0 \ (\mathrm{mod}\ 2)
\end{aligned}$$

Using the basis of $L_{B_{m,n}}$, one can efficiently compute the (Hermite normal) basis of more complex boards. The algorithm consists in covering a board B by overlapping rectangular sub-boards B_i, $i = 1, \ldots, I$. Then we append each matrix $A^i_{m_i,n_i}$ $i = 1, \ldots, I$ consisting of the (row and column) moves within each sub-board B_i and add the *cross moves*, that is, the moves from one sub-board to another. Since the number of cross moves is, in general, quite small, the computation of the resulting (Hermite normal) basis of the initial board B is, in general, a bit tedious but easy. This algorithm also provides a rough upper bound for the determinant, that is, $\det(L_B) \leq \prod_{i=1}^{I} \det(L_{B_i}) \leq 2^{4I}$. For example, the board B_{60} given in Fig. 5 is made of two overlapping square boards $B_{6,6}$ and $B_{5,5}$ with 4 cross-moves centered on the common hole. It directly gives the following proposition which, in particular, excludes any complementary game $(c, \bar{c})$ while, B_{60} being a null-class board, complementary games satisfy the rule-of-three.

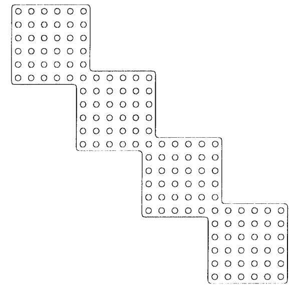

Fig. 6 A hook board B with associated lattice satisfying $\det(L_B) = 128$.

Proposition 4.5: $L_{B_{60}}$ has full rank with determinant $\det(L_{B_{60}}) = 32$, hence $L_{B_{60}} \neq \mathrm{Ker}(\phi)$, and $\boldsymbol{l} \notin L_{B_{60}}$, that is, any complementary game $(c, \bar{c})$ is infeasible on B_{60}.

The computation the basis of $L_{B_{60}}$ can be easily extended to any set of k rectangular boards pairwise overlapping on a common corner. The resulting k-hook board—see for example Fig. 6—will give a lattice with determinant 2^{k+3}. Hook boards demonstrate that the solitaire lattice criterion can be *exponentially finer* than the rule-of-three in the sense of the following theorem.

Theorem 4.6: For hook boards, the solitaire lattice condition exponentially outperforms the rule-of-three, that is, for every k and every k-hook board B, the ratio of the number of congruence classes of $\boldsymbol{Z}^B$ modulo L_B to the number of congruence classes of $\boldsymbol{Z}^B$ modulo $\mathrm{Ker}(\phi)$ satisfies

$$\frac{|\boldsymbol{Z}^B/L_B|}{|\boldsymbol{Z}^B/\mathrm{Ker}(\phi)|} = 2^{k-1}.$$

5. Conclusion

The solitaire game provides an nice application for clas-

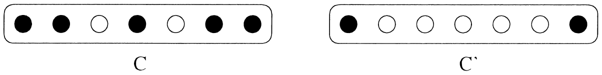

Fig. 7 A lattice and fractional feasible but integer infeasible game.

sical combinatorial tools such as cone, lattice and integer cone. Checking membership in the lattice L_B is usually easy (once we have a basis) and checking membership in the cone C_B amounts to solve a linear program in polynomial time. Combining these two criteria, that is, checking membership in $C_B \cap L_B$, is usually efficient in proving infeasibility. For example, while the game in Fig. 3 satisfies $c' - c \in L_B$ but $c' - c \notin C_B$, the central game (see Fig. 1) played on a French board (an English board with 4 additional holes in positions $(\pm 2, \pm 2)$) satisfies $c' - c \in C_B$ but $c' - c \notin L_B$. Note that for both the French and English boards, we have $L_B = \mathrm{Ker}(\phi)$; therefore checking the membership in L_B can be easily done using the rule-of-three. The membership in C_B of the central game played on a French board can be shown by moving all pegs on the boundary to the inner part of the board, then moving one peg in the center and finally remove the other pegs using the fractional move given after Prop. 3.1. Clearly we have $C_B \cap L_B \subset IC_B$ but this inclusion is strict as illustrated by Fig. 7. A further step could be to find a relatively small generating set (Hilbert basis) for the integer cone IC_B of some interesting classes of boards B.

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